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# A ship's waves and its wake

By D. H. PEREGRINE
School of Mathematics, University of Bristol

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Waves generated at the stern of a ship must travel through the ship's wake. The effect of the mean flow in the wake refracting the waves is calculated by using a much simplified model. It is found that the waves diverging from the stern of a ship may differ considerably from the bow waves, in qualitative agreement with observation.

## 1. Introduction

Prominent components of the wave pattern generated by fine-form ships at low Froude numbers are the waves that diverge from the bow and stern. It has often been remarked that the stern waves are much smaller than the bow waves. The usual theory, linearized potential flow past a thin ship, indicates that the bow and stern waves should be of comparable size. It has been suggested, several times, that this discrepancy is in some way due to the boundary layer and wake of the ship. Gadd (1969), in a recent account of ship wave-making theory which includes comparison with experiments on models, shows how waves generated by a ship may be divided into bow and stern waves, but is only able to follow this up by introducing an empirical reducing factor for the stern waves.

Tatinclaux (1970) has considered the effect of an 'inviscid' wake behind an ogive. The wake velocities were assumed to be sufficiently small so that the problem could be linearized. Even so, for some Froude numbers, the effect is quite appreciable. In the example he gives, with a maximum wake velocity $1/10\pi$ of the forward velocity the wave-making resistance is from 10 % larger to 35 % smaller than for irrotational flow. (The aspect ratio of the ogive in the examples is not given.)

There are several different ways in which the wake may, and probably does, affect the waves but only one is considered here. If the waves are taken to be generated at the ship they must pass through the boundary layer and wake to the otherwise undisturbed water outside. The particular effect considered is the refraction of waves by the mean flow of the wake. This is shown to give a pattern of diverging waves differing from the Kelvin ship wave pattern, for a sufficiently strong wake.

The problem is idealized by considering a wave-making source moving with uniform velocity in the middle of a wake. The source represents the stern of a ship, the assumption being that the after-most part of the ship is more important for wave generation than the smoothly converging region in front of it. If the waves generated diverge at an appreciable angle to the centre-line of the ship, then they will only pass through a small portion of the wake, so that we model this portion by extending the wake uniformly along its length.

The method used is the simplest that shows the basic elements of the Kelvin ship wave pattern when used for still water. Waves of all frequencies are considered to be generated by the source and the envelope of those waves that can remain steady relative to the source is found. Waves on this envelope are assumed to be dominant in the wave pattern. Except for very weak wakes they do not diverge at the Kelvin angle but at a smaller angle to the centre-line of the wake. The form of the waves radiated to infinity depends only on the maximum value of the velocity in the wake. The results are qualitative rather than quantitative; however, there is no restriction to small wake velocities. A limitation of the theory is that it is based on a short wavelength approximation.

## 2. Theory

Introduce a co-ordinate system with $Oy$ along the centre of the wake, $Oz$ vertically upwards and $Ox$ to complete a right-handed set of axes. The wave-making source has velocity $(0, V, 0)$ along $Oy$. The wake is represented by a velocity distribution $(0, U(x,z,), 0)$ with $0 \leqslant U(x,z) \leqslant V$ and $U = 0$ outside a region $-b \leqslant x \leqslant b$, as indicated in figure 1 which represents the plane $z = 0$.

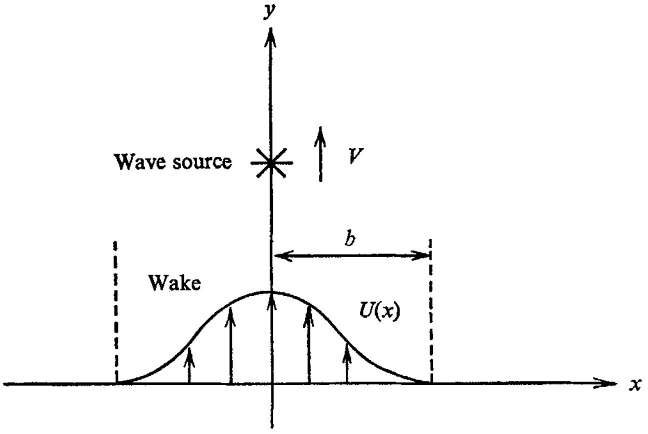


FIGURE 1

The propagation of short gravity waves on non-uniform currents such as this has been treated by Longuet-Higgins & Stewart (1961). They derive expressions for the change in the properties of a wave train as it progresses across a flow where the length scale of the flow is much greater than a wavelength. More recently Smith (1970) has developed another approach to short surface waves which allows non-uniform flows to be included. Smith's technique provides for a systematic expansion in inverse powers of the wave-number and so may be carried to a higher degree of approximation than Longuet-Higgins & Stewart's results. The two methods agree to the first approximation, which is used here, and

Longuet-Higgins & Stewart give explicit expressions. If Smith's technique is used it is clearly seen that it is only the velocity $U(x)$ on $z = 0$ that affects the waves to this approximation.

Consider the waves generated by the source after they have traversed the wake and are propagating in the still water outside. If they are to form part of a steady wave pattern moving with velocity $V$ individual wave components must satisfy the relation

$$c = V \sin\theta, \tag{1}$$

where $c$ is their phase velocity and $\theta$ the angle their direction of propagation makes with the $x$ axis.

Within the wake the wave energy is propagated at a velocity which is the vector sum of the group velocity, which is perpendicular to the wave crests, and the velocity $U(x)$ of the water. If the subscript 1 is used to denote values of wave properties inside the wake and no subscript for the properties of the same waves outside the wake, then Longuet-Higgins & Stewart (1961, equations (8.9)) give us

$$c_1(x) = \frac{c^2}{c - U(x)\sin\theta}$$

and

$$\sin\theta_1(x) = \frac{c^2 \sin\theta}{[c - U(x)\sin\theta]^2}.$$

If we introduce

$$W(x) = 1 - U(x)/V$$

and use equation (1) these may be written more simply as

$$c_1(x) = c/W(x), \quad \text{and} \quad \sin\theta_1(x) = \sin\theta/W^2(x).$$

For deep water the group velocity is equal to half the phase velocity, thus within the wake a wave packet has a velocity

$$\left(\frac{V\sin\theta[W^4(x) - \sin^2\theta]^{\frac{1}{2}}}{2W^3(x)},\, U(x) + \frac{V\sin^2\theta}{2W^3(x)}\right). \tag{2}$$

Now consider the source to have constant velocity $V$ and suppose it is at the origin at $t = 0$. Then the waves emitted at time $-t$ when the source was at $(0, -Vt)$ which satisfy the relation (1) have a locus in the $x, y$ plane which is given by

$$\left.\begin{aligned} x &= b + \tfrac{1}{2}Vt\sin\theta\cos\theta - \int_0^b \frac{W^3(\xi)\cos\theta\, d\xi}{[W^4(\xi) - \sin^2\theta]^{\frac{1}{2}}}, \\ y &= -Vt + \tfrac{1}{2}Vt\sin^2\theta + \int_0^b \frac{[(1 - W^3)\sin^2\theta + 2(1 - W)\,W^3]\, d\xi}{\sin\theta[W^4(\xi) - \sin^2\theta]^{\frac{1}{2}}}, \end{aligned}\right\} \tag{3}$$

for that part of the locus outside the wake with $x > b$. Equations (3) are derived in an appendix, and an example of such a locus is given in figure 2. (For comparison with the still water case, take $b = 0$.)

The predominant part of the wave pattern is given by the envelope of these curves obtained for different values of $t$. This is most easily found by first eliminating $t$. The resulting equation and its partial derivative with respect to $\theta$ then give us the envelope

$$\left.\begin{aligned} x &= b+\int_0^b \frac{W^4(\xi)\cos^3\theta[3\sin^2\theta-2W^4(\xi)]\,d\xi}{(2-3\sin^2\theta)\,[W^4(\xi)-\sin^2\theta]^{\frac{3}{2}}},\\ y &= \int_0^b \frac{[1-W^4(\xi)]\sin^3\theta[3\sin^2\theta-2-2W^4(\xi)]\,d\xi}{(2-3\sin^2\theta)\,[W^4(\xi)-\sin^2\theta]^{\frac{3}{2}}}. \end{aligned}\right\} \tag{4}$$

This curve could be determined by numerical integration for a particular $W(x)$ and the value of $\theta$ at a point on the curve would give the direction of wave crests there.

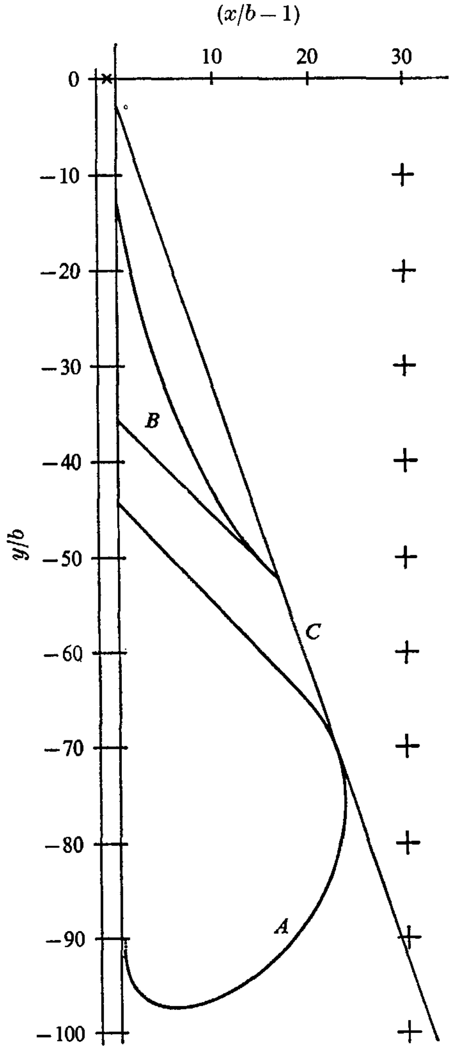


FIGURE 2. Uniform wake: $U = 0{\cdot}12V$, $W^4 = 0{\cdot}6$. The two vertical lines represent the wake boundary. $A$, locus of waves satisfying $c = V\sin\theta$ and generated by the moving source at $y/b = -100$; $B$, lines of constant phase; $C$, envelope of lines like $A$ and $B$.

## 3. Results

There is little value in performing detailed integrations of (4). The general features of the waves can be ascertained fairly simply. For example, if $W(x) = W$ a constant for $|x| < b$ the integration is trivial and

$$\left.\begin{aligned} x &= b + \frac{bW^4\cos^3\theta(3\sin^2\theta - 2W^4)}{(2-3\sin^2\theta)(W^4-\sin^2\theta)^{\frac{3}{2}}}, \\ y &= \frac{b(1-W^4)\sin^3\theta(3\sin^2\theta - 2 - 2W^4)}{(2-3\sin^2\theta)(W^4-\sin^2\theta)^{\frac{3}{2}}}. \end{aligned}\right\} \tag{5}$$

Figure 2 shows this envelope for $W^4 = 0{\cdot}6$, i.e. $U = 0{\cdot}12V$.

The behaviour of these waves as they radiate towards infinity is easily found. For $x, y$ in (5) to approach infinity the denominator on the right-hand side of each equation must approach zero. There are two possible ways in which they may happen, depending on the value of $W$.

If $W^4 > \frac{2}{3}$ then the denominator is zero if

$$2 - 3\sin^2\theta = 0. \tag{6}$$

This is the same value of $\theta$ as for the Kelvin ship wave pattern and corresponds to a 'weak' wake; $W^4 > \frac{2}{3}$ implies $U < 0{\cdot}1V$.

The case $W^4 < \frac{2}{3}$ is of more interest since the denominators in (5) are then zero when

$$\sin^2\theta = W^4. \tag{7}$$

This gives a smaller value of $\theta$ than for the Kelvin pattern. The envelope (5) has as its asymptotes the line

$$y = -\frac{1+\cos^2\theta}{\sin\theta\cos\theta}x, \tag{8}$$

where $\theta$ takes the value given by (7). These lines are at a smaller angle to the direction of motion than the corresponding lines of the Kelvin ship wave pattern. [It is easy to show that if a straight line of waves with crests at an angle $\theta$ to the $y$ axis is advancing in the $y$ direction with velocity $V$ and (1) is satisfied then (8) is the equation of such a line.] The total range of $\theta$ is not great since at $x = b$, $\sin^2\theta = \frac{2}{3}W^4$ so the line of the diverging waves is close to a straight line, as may be seen in figure 2.

Returning to the general case, equations (4), the behaviour of waves at infinity is again easy to see. For a 'weak' wake, that is, $W(x)$ always $> \frac{2}{3}$, we have (6) again. For a 'stronger' wake, where the minimum value of $W^4(x) < \frac{2}{3}$ the integral will diverge unless

$$\sin^2\theta < \min\{W^4(x)\}, \tag{9}$$

giving us the value of $\theta$ for $x, y \to \infty$. The asymptotes to the envelope will again be given by (8). It is interesting that the behaviour of the wave envelope at infinity is governed by the maximum velocity in the wake (i.e. minimum of $W(x)$) in this model theory, and not by any other property of the wake.

In the latter case of a 'strong' wake the wavelength of the waves may be considerably shorter than for a 'weak' wake, so that the short wave approach used here is more likely to be relevant. The variation of wavelength of the waves at great distances with $U/V$ is given in figure 3 which, for example, shows the wavelength to be reduced by half for $U/V = 0{\cdot}25$. This also means that the amplitude of the waves is likely to be less in about the same proportion since a wave train can only have a certain maximum steepness without breaking, and hence the energy and momentum radiated in the waves from the stern of a vessel is likely to be considerably less than in the bow waves. The smaller angle of divergence also implies a reduced transport of momentum away from the ship.

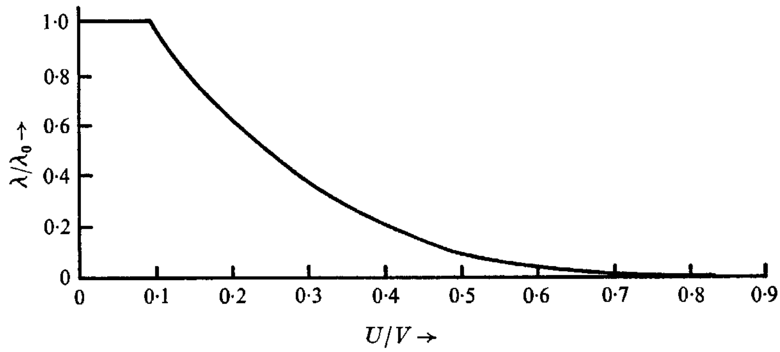


FIGURE 3. Wavelength of diverging waves as $x, y \to \infty$ for different wakes. $\lambda_0$ = wavelength with no wake, $U$ = maximum velocity of wake.

The relevance of this theory to a ship or model depends mainly on the width and structure of the wake compared with the wavelength of the waves being generated. If the width of the wake is less than a wavelength it is unlikely that this approach would yield any benefit. However, by measuring the stern waves it would be possible to evaluate a 'wake parameter', corresponding to the maximum value of $U(x)/V$ in this theory, that may be of some use.

## 4. Comparison with observations

There are few observations on the pattern of ship waves and few measurements of wake velocities. In some photographs of ships the stern waves may be seen clearly to have smaller values of $\theta$ than the bow waves.

The pattern of ship waves for a range of ship models has been measured by Kajitani (1963, 1965). Several diagrams show diverging stern waves clearly and for most of them $\theta$ is clearly less than it is for the bow waves. For example, model S-201 at Froude number 0·267 (Kajitani 1965, figure 29, page 105) has $\theta = 30^\circ$ and the line of wave crests at 13° to the line of motion for the stern waves, though the corresponding figures for the bow waves are 44° and 20°, which is not in agreement with the Kelvin ship wave values. It is not possible to measure these angles very accurately or very far from the model; however, the difference between the two sets of waves is clear. The above value for $\theta = 30^\circ$ corresponds to a minimum value of $W^2(x)$ of 0·5 or to a maximum value of $U$ of $0{\cdot}3V$.

The only measurement of wake velocities both near the surface and close to the stern of the model that I have been able to find are by Hogben (1964) for a 'mathematical' model with parabolic waterlines and sections. The measurements were made 1 in. (2·5 cm) below the surface at the stern of the model which was 20 ft (6 m) long and 2 ft (60 cm) wide. Measurements are given for Froude numbers of 0·156 and 0·315. They show maximum values of $U/V$ of 0·84 and 0·73 respectively. However the width of the wake is small, $U/V > 0{\cdot}1$ for about 5 in. (13 cm) each side of the centre-line. The wavelengths of diverging waves in the corresponding Kelvin ship wave patterns are 8 in. (20 cm) and 15 in. (38 cm) respectively. Thus if this theory provides any guide at all in this case the waves are likely to correspond to quite a small value of the 'wake parameter' $U/V$.

## 5. Conclusion

The approach used here is too simple to apply directly to a ship. However, it does show that the shear flow around the stern of a vessel can have a substantial qualitative effect on the waves generated. The form of wave pattern predicted is in agreement with observation. From observation of the diverging stern waves it is possible to work out a corresponding value of maximum $U/V$ in the wake. It would be interesting to have measurements of both wave patterns and wake profiles to see how this value might compare with actual values of $U/V$.

I wish to thank Dr G. E. Gadd for stimulating discussions at the National Physical Laboratory, Ship Division.

## Appendix

If $T$ is the time taken for a wave packet to reach the edge of the wake at $x = b$, then

$$T = \int_{x=0}^{x=b} dt = \int_0^b \frac{dt}{dx}\,dx = \int_0^b \frac{2W^3(x)\,dx}{V\sin\theta[W^4(x)-\sin^2\theta]^{\frac{1}{2}}} \qquad \text{(A 1)}$$

from expression (2).

Similarly the distance the wave packet has travelled relative to still water in the $y$ direction is

$$\int_{x=0}^{x=b} dy = \int_0^b \frac{dy}{dt}\frac{dt}{dx}\,dx = \int_0^b \left[U(x) + \frac{V\sin^2\theta}{2W^3(x)}\right] \frac{2W^3(x)\,dx}{V\sin\theta[W^4(x)-\sin^2\theta]^{\frac{1}{2}}} = y_b. \qquad \text{(A 2)}$$

For $t > T$ the wave packet has velocity $(\frac{1}{2}V\sin\theta\cos\theta, \frac{1}{2}V\sin^2\theta)$ and thus its position at $t = 0$ when it was emitted at $(0, -Vt)$ at time $-t$ is

$$\left.\begin{aligned} x &= b + \tfrac{1}{2}V(t-T)\sin\theta\cos\theta, \\ y &= y_b - Vt + \tfrac{1}{2}V(t-T)\sin^2\theta. \end{aligned}\right\} \qquad \text{(A 3)}$$

Substitution of (A 1) and (A 2) then gives equations (3).

Lines of constant phase are those for which

$$\chi = \mathbf{k}.\mathbf{r} - \sigma t$$

is constant. By using $c = V\sin\theta = \sigma/k = g/\sigma$ we find

$$\chi = \frac{g}{V^2\sin^2\theta}(\cos\theta x + \sin\theta y - \sin\theta Vt).$$

Substitution from equations (3) for $x$ and $y$ leads to a relation between $t$, $\theta$ and $\chi$ which together with equations (3) has been used to calculate lines of constant phase such as that shown in figure 2, where $\chi = -150gb/V^2$.

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